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On Some Applications of New Integral Transform "ELzaki Transform"

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Abstract

In this study a new integral transform, namely ELzaki transform was applied to solve linear ordinary differential equations with constant coefficients. In particular we apply this new transform technique to solve linear dynamic systems and signals-delay differential equations and the renewal equation in statistics.

Keywords: Elzaki Transform- Differential Equations--Applications.

Introduction

Many problems of physical interest are described by differential and integral equations with appropriate initial or boundary conditions. These problems are usually formulated as initial value problem, boundary value problems, or initial – boundary value problem that seem to be mathematically more vigorous and physically realistic in applied and engineering sciences. ELzaki transform method is very effective for solution of differential and integral equations.

The technique that we used is ELzaki transform method which is based on Fourier transform, it introduced by Tarig Elzaki (2010) see [1, 2, 3, 4, 5, 6].

In this study, ELzaki transform is applied to solve linear dynamic systems and signals-delay differential equations and the renewal equation in statistics, which the solution of these equations have a major role in the fields of science and engineering.

When a physical system is modeled under the differential sense, if finally gives a differential equation.

Recently .Tarig ELzaki introduced a new transform and named as ELzaki transform [1] which is defined by:

$$\mathrm{E}\left[f(t),u\right]=T(u)=u\int_0^\infty e^{-\frac{t}{u}}f(t)\,dt \quad , \quad u\in(k_1,k_2)$$

Or for a function $f(t)$ which is of exponential order,

$$\left|f(t)\right|<\begin{cases} Me^{-t/k_1} \quad , \quad t\le 0 \\ Me^{t/k_2} \quad , \quad t\ge 0 \end{cases}$$

$$\mathrm{E}\left[f(t),u\right]=T(u)=u^2\int_0^\infty e^{-t}f(ut)\,dt \quad , \quad u\in(k_1,k_2)$$

Theorem (1)

Let $T(u)$ is the ELzaki transform of $\left[E\left(f(t)\right)=T(u)\right]$. then:

$$(i) \quad \mathrm{E}\left[f'(t)\right]=\frac{T(u)}{u}-u\,f(0) \qquad (ii) \quad \mathrm{E}\left[f''(t)\right]=\frac{T(u)}{u^2}-f(0)-uf'(0)$$

$$(iii) \quad \mathrm{E}\left[f^{(n)}(t)\right]=\frac{T(u)}{u^n}-\sum_{k=0}^{n-1}u^{2-n+k}f^{(k)}(0)$$

Proof

(i) $\mathrm{E}\left[f'(t)\right]=u\int_0^\infty f'(t)\, e^{\frac{-t}{u}}\,dt$ Integrating by parts to find that:

$$\mathrm{E}\left[f'(t)\right]=\frac{T(u)}{u}-u\,f(0)$$

(ii) Let $g(t)=f'(t)$, then: $\mathrm{E}\left[g'(t)\right]=\frac{1}{u}\,\mathrm{E}\left[g(t)\right]-ug(0)$

We find that, by using (i):

$$\mathrm{E}\left[f''(t)\right] = \frac{T(u)}{u^2}-f(0)-uf'(0)$$

(iii) Can be proof by mathematical induction

Theorem (2)

Let $f(t) \in A = \left\{ f(t) \middle| \exists M, k_1, k_2 > 0, such \;\; that \left| f(t) \right| < Me^{|t|/k_i} \; , \; if \; t \in (-1)^j \times [0, \infty) \right\}$

With Laplace transform $F(s)$, Then ELzaki transform $T(u)$ of $f(t)$ is given by $T(u) = u\,F\left(\frac{1}{u}\right)$

Proof

Let: $f(t) \in A$, then for $-k_1 < u < k_2$ $T(u) = u^2 \int_0^\infty e^{-t} f(ut)\,dt$

Let $w = ut$ then we have: $T(u) = u^2 \int_0^\infty e^{-\frac{w}{u}} f(w) \frac{dw}{u} = u \int_0^\infty e^{-\frac{w}{u}} f(w)\,dw = u\,F\left(\frac{1}{u}\right).$

Also we have that $T(1) = F(1)$ so that both the ELzaki and Laplace transforms must coincide at $u = s = 1$

Theorem (3)

Let $f(t)$ *and* $g(t)$ be defined in A having Laplace transforms $F(s)$ and $G(s)$ and ELzaki transforms $M(u)$ and $N(u)$. Then ELzaki transform of the convolution of f and g

$(f*g)(t) = \int_0^\infty f(t) g(t-\tau)\,d\tau$ Is given by: $E\left[(f*g)(t)\right] = \frac{1}{u} M(u) N(u)$

Proof

The Laplace transform of $(f * g)$ is given by: $L\left[(f*g)\right] = F(s)G(s)$

By the duality relation (Theorem (2)), we have: $E\left[(f*g)(t)\right] = u\,L\left[(f*g)(t)\right]$, and since

$$M(v) = u\,F\left(\frac{1}{u}\right) \; , \; N(u) = u\,G\left(\frac{1}{u}\right) \; \textit{Then} \; E\left[(f*g)(t)\right] = u\left[F\left(\frac{1}{u}\right).G\left(\frac{1}{u}\right)\right]$$

$$u\left[\frac{M(u)}{u} . \frac{N(u)}{u}\right] = \frac{1}{u} M(u) N(u)$$

Theorem (4)

If $\mathrm{E}\left[f(t)\right] = T(u)$, then: $\mathrm{E}\left[e^{-at} f(t)\right] = (1+au) T\left(\frac{u}{1+au}\right)$

Where a is a real constant.

Proof

We have, by definition $E\left[e^{-at}f\left(t\right)\right]=u\int_0^\infty e^{-\frac{(1+au)}{u}t}f\left(t\right)dt$. Let $w=\frac{u}{1+au}$ or $u=\frac{w}{1-aw}$, we have:

$$u\int_0^\infty e^{-\frac{a}{u}}f\left(t\right)dt=\frac{w}{1-aw}\int_0^\infty e^{-\frac{t}{w}}f\left(t\right)dt=\frac{1}{1-aw}T\left(w\right)=\frac{1}{1-\frac{au}{1+au}}T\left[\frac{u}{1+au}\right] \quad and$$

$$E\left[e^{-at}f\left(t\right)\right]=\left(1+au\right)T\frac{u}{1+au}$$

Theorem (5)

If $E\left[f\left(t\right)\right]=T\left(u\right)$, then: $E\left[f\left(t-a\right)H\left(t-a\right)\right]=e^{-\frac{a}{u}}T\left(u\right)$

Where $H\left(t-a\right)$ is Heaviside unit step function.

Proof

It follows from the definition that:

$$E\left[f\left(t-a\right)H\left(t-a\right)\right]=u\int_0^\infty e^{-\frac{a}{u}}f\left(t-a\right)H\left(t-a\right)dt=u\int_0^\infty e^{-\frac{t}{u}}f\left(t-a\right)dt$$

Let $t-a=\tau$, then we have: $e^{-\frac{a}{u}}u\int_0^\infty e^{-\frac{t}{\tau}}f\left(\tau\right)d\tau=e^{-\frac{a}{u}}T\left(u\right)$

In particular if $f\left(t\right)=1$, then: $E\left[H\left(t-a\right)\right]=u^2e^{-\frac{a}{u}}$

Also we can prove by mathematical induction that: $E\left[\frac{\left(t-a\right)^{n-1}}{\Gamma\left(n\right)}H\left(t-a\right)\right]=u^{n+1}e^{-\frac{a}{u}}$

Example (1) (Linear dynamical systems and signals)

In physical and engineering sciences, a large number of linear dynamical systems with a time dependent input signal $f\left(t\right)$ that generates an output signal $x\left(t\right)$ can be described by the ordinary differential equation with constant coefficients.

$$\left(D^n+a_{n-1}D^{n-1}+...+a_0\right)x\left(t\right)=\left(D^m+b_{m-1}D^{m-1}+....+b_0\right)f\left(t\right) \qquad (1)$$

Where $D=\frac{d}{dx}$, $a_0,a_1,...a_{n-1}$, $b_0,b_1,...b_{m-1}$ are constants

We apply ELzaki transform to find the output $x\left(t\right)$ so that (1) becomes.

$$\bar{P}_n\left(u\right)\bar{x}\left(u\right)-\bar{R}_{n-1}\left(u\right)=\bar{q}_m\left(u\right)\bar{f}\left(u\right)-\bar{s}_{m-1}\left(u\right) \qquad (2)$$

Where,

$$\bar{P}_n(u)=\frac{1}{u^n}+\frac{a_{n-1}}{u^{n-1}}+\ldots+a_0 \quad , \quad \bar{q}_m(u)=\frac{1}{u^m}+\frac{b_{m-1}}{u^{m-1}}+\ldots+b_0$$

$$\bar{R}_{n-1}(u)=\sum_{k=0}^{n-1}u^{2-n+k}x^{(k)}(0)+a_{n-1}\sum_{k=0}^{n-2}u^{3-n+k}x^{(k)}(0)+\ldots+ux(0)$$

$$\bar{S}_{m-1}(u)=\sum_{k=0}^{m-1}u^{2-m+k}f^{(k)}(0)+b_{m-1}\sum_{k=0}^{m-2}u^{3-m+k}f^{(k)}(0)+\ldots+uf(0)$$

It is convenient to express (2) in the form

$$\bar{x}(u)=\bar{h}(u)\bar{f}(u)+\bar{g}(u) \tag{3}$$

Where

$$\bar{h}(u)=\frac{\bar{q}_m(u)}{\bar{p}_n(u)} \text{ And } \bar{g}(u)=\frac{\bar{R}_{n-1}(u)-\bar{S}_{m-1}(u)}{\bar{P}_n(u)} \tag{4}$$

And $\bar{h}(u)$ is usually called the transfer function. The inverse ELzaki transform combined with the convolution theorem leads to the formal solution,

$$x(t)=\int_0^t f(t-\tau)h(\tau)\,d\tau+g(t) \tag{5}$$

With zero initial data, $\bar{g}(u)=0.$ the transfer function takes the simple form.

$$\bar{h}(u)=\frac{\bar{x}(u)}{\bar{f}(u)} \text{ or } \bar{x}(u)=\bar{h}(u)\ \bar{f}(u) \tag{6}$$

If $f(t)=\delta(t)$ and $h(t)=e^t$ then, the output function is

$x(t)=E^{-1}\left[\frac{u^3}{1+u}\right]=e^t-1$ $h(t)$ is known as the impulse response.

Example (2) (Delay Differential Equations)

In many problems the derivatives of the unknown function $x(t)$ are related to its value at different times $t-\tau$.this leads us to consider differential equations of the form:

$$\frac{dx}{dt}+ax(t-\tau)=f(t) \tag{7}$$

Where a is a constant and $f(t)$ is a given function. Equations of this type are called delay differential equations. In general , initial value problems for these equations involve the specification of $x(t)$ in the interval $t_0-\tau\le t\le t_0$ and this information combined with the equation it self, is sufficient to determine $x(t)$ for $t > t_0$. We show how equation (7) can be solved by ELzaki transform when $t_0 = 0$ and $x(t) = x_0$ for $t \le 0$. In view of the initial condition , we can write.

$x(t-\tau) = x(t-\tau)H(t-\tau)$ So equation (7) is equivalent to,

$$\frac{dx}{dt} + ax(t-\tau)H(t-\tau) = f(t) \tag{8}$$

Applying ELzaki transform to (8) gives, $\dfrac{\bar{x}(u)}{u} - ux(0) + ae^{-\frac{\tau}{u}}\bar{x}(u) = \bar{f}(u)$

Or $\bar{x}(u) = \dfrac{u\bar{f}(u) + u^2x(0)}{1 + aue^{-\frac{\tau}{u}}} = \left[u\,\bar{f}(u) + u^2x(0)\right]\left[1 + aue^{-\frac{\tau}{u}}\right]^{-1}$

And

$$\bar{x}(u) = \left[u\bar{f}(u) + u^2x(0)\right]\sum_{n=0}^{\infty}(-1)^n(au)^n e^{-\frac{n\tau}{u}} \tag{9}$$

The inverse ELzaki transform gives the formal solution:

$$x(t) = \mathrm{E}^{-1}\left[u\,\bar{f}(u) + u^2x(0)\right]\sum_{n=0}^{\infty}(-1)^n(au)^n e^{-\frac{n\tau}{u}} \tag{10}$$

In order to write an explicit solution, we choose $x_0 = 0$ and $f(t) = t$ and hence (10) be comes.

$$x(t) = \mathrm{E}^{-1}\left[u^4\sum_{n=0}^{\infty}(-1)^n(au)^n e^{-\frac{n\tau}{u}}\right] = \sum_{n=0}^{\infty}(-1)^n a^n \frac{(t-n\tau)^{n+2}}{(n+2)!}H(t-n\tau) \quad , \quad t > 0$$

Example (3) (Renewal Equation in statistics)

The random function $x(t)$ of time t represents the number of times some event has occurred between time 0 and time t, and usually referred to as a counting Process. A random variable X_n that recodes the time it assumes for X to get the value n from the $n-1$ is referred to as an inter-arrival time .If the random variables $X_1, X_2 ....$ are independent and identically distributed, then the counting process $X(t)$ is called a renewal process. We denote their common Probability distribution function by $F(t)$ and the density function by $f(t)$ so that $F'(t) = f(t)$.Then renewal function is defined by the expected number of time the event being counted occurs by time t and is denoted by $r(t)$ so that .

$$r(t) = \mathrm{E}\left[X(t)\right] = \int_0^\infty \mathrm{E}\left\{\left[x(t)\right] : X_1 = x\right\} f(x)\, dx \qquad (11)$$

Where $\mathrm{E}\left\{X(t) : X_1 = x\right\}$ is the conditional expected value of $X(t)$ under the condition that $X_1 = x$ and has the value.

$$\mathrm{E}\left\{X(t) : X_1 = x\right\} = \left[1 + r(t-x)\right] H(t-x) \qquad (12)$$

Thus

$$r(t) = \int_0^t \left[1 + r(t-x)\right] f(x)\, dx$$

Or

$$r(t) = F(t) + \int_0^t r(t-x) f(x)\, dx \qquad (13)$$

This is called the renewal equation in mathematical statistics .We solve the equation by taking ELzaki transform with respect to t, and ELzaki transformed equation is,

$$\bar{r}(u) = \bar{F}(u) + \frac{1}{u}\bar{r}(u)\bar{f}(u) \text{ Or } \bar{r}(u) = \frac{u\bar{F}(u)}{u - \bar{f}(u)}$$

The inverse transform gives the formal solution $r(t)$ of renewal function.

(i) If $F(t) = e^t$ and $f(t) = e^t$, then: $r(t) = \mathrm{E}^{-1}\left[\dfrac{u^3}{1-2u^2}\right] = \dfrac{1}{\sqrt{2}} \sin\mathrm{h}\sqrt{2}t$

(ii) If $F(t) = t$ and $f(t) = 1$, then: $r(t) = E^{-1}\left[\dfrac{u^2}{1-u} - u^2\right] = e^t - 1$

Conclusion

In this study, we introduced new integral transform called ELzaki transform to solve linear dynamical systems and signals – delay differential equations and the renewal equation in statistics. It has been shown that ELzaki transform is a very efficient tool for solving these equations

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Appendix

ELzaki Transform of some Functions

$f(t)$	$E[f(t)] = T(u)$
1	u^2
t	u^3
t^n	$n!u^{n+2}$
$t^{a-1}/\Gamma(a), a>0$	u^{a+1}
e^{at}	$\frac{u^2}{1-au}$
te^{at}	$\frac{u^3}{(1-au)^2}$
$\frac{t^{n-1}e^{at}}{(n-1)!}$, $n=1,2,....$	$\frac{u^{n+1}}{(1-au)^n}$
$\sin at$	$\frac{au^3}{1+a^2u^2}$
$\cos at$	$\frac{u^2}{1+a^2u^2}$
$\sinh at$	$\frac{au^3}{1-a^2u^2}$
$\cosh at$	$\frac{au^2}{1-a^2u^2}$
$e^{at}\sin bt$	$\frac{bu^3}{(1-au)^2+b^2u^2}$
$e^{at}\cos bt$	$\frac{(1-au)u^2}{(1-au)^2+b^2u^2}$
$t\sin at$	$\frac{2au^4}{1+a^2u^2}$
$J_0(at)$	$\frac{u^2}{\sqrt{1+au^2}}$
$H(t-a)$	$u^2e^{-\frac{a}{u}}$
$\delta(t-a)$	$ue^{-\frac{a}{u}}$